and sgn denotes the signum function.† The first sum in Eq. (4a) holds for those integral values of n for which $\Gamma > 0$, and the second sum for those n for which $\Gamma < 0$. n varies from $(-\infty)$ to $(+\infty)$. As $|n| \to \infty$, $\Gamma \to (2\pi n)^2 (1-M^2)$, and in the subsonic case Γ is positive. Thus, when $M<1$, the contributions to the infinite sum from terms for which $|n|$ is large are exponentially small. Thus Eq. (4a) is a rapidly (i.e., exponentially fast) converging series; this series is quite suitable for numerical computation. Second, Eq. (4a) is valid even for supersonic flows ($M>1$) as long as $M\sin\lambda < 1$. This case is of interest in modern compressors that operate at supersonic relative Mach numbers but whose axial Mach number is less than unity. However, in the latter situation, Γ is negative as $|n| \to \infty$ and Eq. (4a) is less suitable for numerical computation because of the large number of terms (or propagating modes) that contribute to the sum. It is possible to extract certain terms from Eq. (4a) to greatly enhance its convergence characteristics in the supersonic case. This will be discussed later.

Observe that Γ is a quadratic function of $(2\pi n - \sigma)$. In the purely subsonic case, this quadratic opens upward and assumes a negative value at the origin. On the other hand, at the point where the sgn function in Eq. (4a) changes sign, Γ is positive. It is obvious then that whenever Γ is negative, so is the sgn function, so that this factor in Eq. (4a) may be replaced by (-1). In this case the trigonometric functions may be combined to form $\exp[-i|\bar{y}|(-\Gamma)^{1/2}/s\beta]$; this exponential together with $\exp(i\omega t)$ forms an outgoing wave in $\bar{y}$ space. That this must be so is clear from the Sommerfeld radiation condition. The situation is quite different in the supersonic case ($M>1$, $M\sin\lambda<1$). After reasoning along the previous lines, it is found that when Γ is negative, sgn will assume both positive and negative values in Eq. (4a). Therefore, the trigonometric functions will combine to form both $\exp[\pm i|\bar{y}|(-\Gamma)^{1/2}/s\beta]$ and the Sommerfeld radiation condition cannot be satisfied in supersonic flows.

Now Eq. (4a) represents the long time ($t \to \infty$) pressure field of an array of pulsating sources that obey Eq. (1). The y derivative of Eq. (4a) gives the pressure field of an array of vertical doublets; it is possible to show after considerable algebra that this y derivative is equivalent to Eq. (34) of Kaji and Okazaki after an obvious change in notation is made. This equivalence is valid in the purely subsonic case. An easier way to establish the correspondence between the current results and those of Ref. 1 is to observe that, apart from some factors of proportionality, I_3 of Kaji and Okazaki gives the pressure field of an array of sources. In fact, the last member of Eq. (19) of Ref. 1 is equivalent to the present result (4a).

In the supersonic leading edge case, the series representation of the pressure given by Eq. (4a) is only conditionally convergent. Nevertheless, it is possible to enhance greatly the convergence characteristics of the above series. Observe that as $|n| \to \infty$,

$$(-\Gamma)^{1/2} \cong \beta|2\pi n|\left\{1-\left[\left(\sigma+\frac{\omega s}{c_\infty}\frac{M\cos\lambda}{\beta^2}\right)\Big/2\pi n\right]+\ldots\right\} \qquad (5)$$

so that by adding and subtracting series of the type

$$\sum_{n=1}^{\infty}\frac{\sin n\mu}{\cos n\nu}\Big/ n^{\epsilon}; \quad \epsilon = 1 \text{ and } 2$$

for suitable choices of μ and ν, Eq. (4a) can be made absolutely convergent with speed n^{-3}. Note that the values of μ and ν may be deduced from Eq. (5) and the asymptotic behavior of the summand in Eq. (4a) as $|n| \to \infty$. It is not known whether the present approach offers an advantage (over, for example, Ref. 5) since numerical calculations have not been carried out as yet.

† $\text{sgn}(x) = \pm 1$ according to x positive or negative.

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Hamilton's Law and the Stability of Nonconservative Continuous Systems

80006

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Introduction

THE law of varying action is applied to obtain direct analytical solutions to nonconservative follower force systems. The solution to the Beck problem[1] is illustrated explicitly because it happens to be well known and is one of the relatively few such systems to which an exact solution may be obtained for comparison of results.

Hamilton's law of varying action,

$$\delta\int_{t_0}^{t_1}(T+W)\,\mathrm{d}t - \Sigma\frac{\partial T}{\partial \dot{q}_i}\delta q_i\Big|_{t_0}^{t_1} = 0 \qquad (1)$$

has now been applied, contrary to statements contained in Refs. 2 and 3, to generate the spacetime path and/or configuration of nonconservative systems for which the work cannot be expressed as a potential function.[4] The application is characterized by the same simplicity and accuracy as demonstrated for other conservative and nonconservative systems.[5-8]

Problem Formulation

When it is specified that the reference coordinate system (shown in Fig. 1) moves with, at most, constant velocity and that there are no discrete particles or rigid bodies associated with the deformable body, the kinetic energy of the body relative to the reference frame is simply $T = \int_V (\rho/2)(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\mathrm{d}V$, where u, v, and w are the displacement components of a material point of the body.

Insight into the meaning of Hamilton's law and Hamilton's principle[5,8] permits the work function to be expressed simply as the sum of the products of the internal and external forces with their respective associated displacements, regardless of what the forces may be a function; in the physical system.[9] For a deformable body with a stress field σ, continuous body forces B, continuous body moments $\bar{M}$, continuous surface forces $\bar{S}$, continuous surface moment M, discrete body and/or surface forces, and discrete body and/or surface moments, the work function (not the work)[9] for application of

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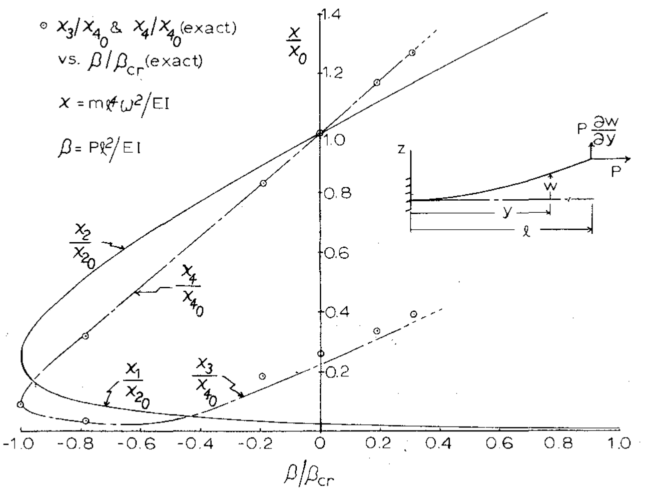

Fig. 1 Frequency ratio vs axial load, lowest four modes.

Hamilton's law is:

$$W = -\int_V (\sigma dA)(\epsilon ds) + \int_V B\cdot r dV + \int_V \bar{M}\cdot d\theta dV$$

$$+ \int_S \bar{S}\cdot r dS + \int_S M\cdot d\theta dS + \Sigma P_i\cdot r_i + \Sigma M_i\cdot d\theta_i \tag{2}$$

It should be noted that none of the terms in this work function is associated with the boundary conditions.

This Note is restricted to systems which may be assumed to conform to engineering beam theory and a general formulation is presented in Ref. 10. For motion restricted to the y-z plane, the assumption that plane sections remain plane yields for the axial component of displacement $v = v_0 - z(\partial w/\partial y)$ and the applicable strain equations are $\epsilon_{yy} = \partial v/\partial y + (\partial w/\partial y)^2/2$ and $\gamma_{yz} = \partial v/\partial z + \partial w/\partial y - (\partial v/\partial z)(\partial w/\partial z) - (\partial v/\partial y)(\partial w/\partial y)$. The stress-strain equations, for the purpose of this paper, are assumed to be linear, $\sigma_{yy} = E\epsilon_{yy}$ and $\tau_{yz} = G\gamma_{yz}$. With these equations and a continuous applied axial load $\bar{S}_y$, discrete applied axial load P_y, continuous applied transverse load $\bar{S}_z$, continuous applied moment $\bar{M}_x$, discrete applied moment M_x, and discrete applied transverse load P_z, Hamilton's law yields two coupled energy equations for the motion of a beam-column system (details are presented in Ref. 10).

One equation is for the axial component of motion.

$$\int_{t_0}^{t_1}\left\{\int_y m\dot{v}_0\delta\dot{v}_0 dy - \int_y EA\frac{\partial v_0}{\partial y}\delta\frac{\partial v_0}{\partial y}dy\right.$$

$$-\int_y \frac{EA}{2}\left(\frac{\partial w}{\partial y}\right)^2\delta\frac{\partial v_0}{\partial y}dy + \int_y \bar{S}_y\delta v_0 dy$$

$$\left. + P_y\delta v_0\right\}dt - \int_y m\dot{v}_0\delta v_0 dy\Big|_{t_0}^{t_1} = 0 \tag{3}$$

and one equation is for the transverse component of motion,

$$\int_{t_0}^{t_1}\left\{\int_y m\dot{w}\delta\dot{w}dy - \int_y EI\frac{\partial^2 w}{\partial y^2}\delta\frac{\partial^2 w}{\partial y^2}dy\right.$$

$$-\int_y EA\frac{\partial v_0}{\partial y}\frac{\partial w}{\partial y}\delta\frac{\partial w}{\partial y}dy + \int_y \bar{S}_z\delta w dy$$

$$\left. + \int_y \bar{M}_x\delta\frac{\partial w}{\partial y}dy + M_x\delta\frac{\partial w}{\partial y} + P_z\delta w\right\}dt - \int_y m\dot{w}\delta w dy\Big|_{t_0}^{t_1} = 0 \tag{4}$$

Solutions to both stationary and nonstationary systems may be obtained from Eqs. (3) and (4).[8] If it is now assumed that $\dot{v}_0$ is negligibly small and that the slope is to be kept small such that the term containing $(\partial w/\partial y)^2$ is also negligibly small, the two equations uncouple. When it is further assumed that the beam-column is uniform and constrained to $v_0 = 0$ at $y = 0$ and that the functional form of the continuous applied axial loads is $\bar{S}_y = Y_1 + Y_2 y$, with a discrete axial load P_y applied at $y = \ell$, Eq. (3) yields the axial component of displacement v_0, from which its derivative is:

$$\frac{\partial v_0}{\partial y} = \frac{\ell}{EA}(Y_0 - Y_1 y - Y_2 y^2/2) \tag{5}$$

where the constant $Y_0 = P_y/\ell + Y_1 + Y_2/2$.

The applied transverse loads and applied moments are now assumed to be functions of the displacment and its derivative

$$\bar{S}_z = Z_1 w + Z_2\frac{\partial w}{\partial y}$$

$$\bar{M}_x = \bar{M}_1 w + \bar{M}_2\frac{\partial w}{\partial y}$$

$$M_x = M_1 w + M_2\frac{\partial w}{\partial y}$$

$$P_z = P_1 w + P_2\frac{\partial w}{\partial y} \tag{6}$$

Prescribing the constants, $P_y, Y_1, Y_2, Z_1, Z_2, \bar{M}_1, \bar{M}_2, M_1, M_2, P_1$, and P_2 orders the magnitude of the various loads. If stationary harmonic motion is possible for this system as defined, it will be given by a stationary harmonic solution

$$w(y,t) = \sum_{i=0}^{N} A_i(\ell - y)^Q y^{i+P}\sin\beta t, \text{ (or } \cos\omega t \text{ or } e^{i\omega t}) \tag{7}$$

When it is further assumed that the end of the beam-column is free at $y = 0$, then Q is also equal to zero.[4] Equation (4) then yields the following eigenvalue equation:

$$\left[\frac{\chi + \beta_3}{i+j+2P+1} - \frac{(i+P)(i+P-1)(j+P)(j+P-1)}{i+j+2P-3}\right.$$

$$-(\beta_0 - \beta_6)\frac{(i+P)(j+P)}{i+j+2P-1} + \beta_1\frac{(i+P)(j+P)}{i+j+2P}$$

$$+\beta_2\frac{(i+P)(j+P)}{i+j+2P+1} + \beta_4\frac{i+P}{i+j+2P} + \beta_5\frac{j+P}{i+j+2P}$$

$$\left. + \beta_7(j+P) + \beta_8(i+P)(j+P) + \beta_9 + \beta_{10}(i+P)\right]\{B_j\}$$

$$= 0, \quad P = 0,1,2 \quad i,j = 0,1,2,\ldots,N \tag{8}$$

The various parameters are defined in terms of the physical quantities of the system, $\chi = m\ell^4\omega^2/EI$;

$$\beta_0 = Y_0\ell^3/EI \qquad Y_0 = P_y/\ell + Y_1 + Y_2/2$$

$$\beta_1 = Y_1\ell^3/EI \qquad \beta_2 = Y_2\ell^3/2EI$$

$$\beta_3 = Z_1\ell^4/EI \qquad \beta_4 = Z_2\ell^3/EI$$

$$\beta_5 = \bar{M}_1\ell^3/EI \qquad \beta_6 = \bar{M}_2\ell^2/EI$$

$$\beta_7 = M_1\ell^2/EI \qquad \beta_8 = M_2\ell/EI$$

$$\beta_9 = P_1\ell^3/EI \qquad \beta_{10} = P_2\ell^2/EI$$

Table 1 Nonconservative beam-column

	Lowest Two Modes					
$P\ell^2/EI$	$m\ell^4\omega_1^2/EI$			$m\ell^4\omega_2^2/EI$		
Terms	3	6	Exact	3	6	Exact
21.0	2.0468	3.4595	3.4593	774.44	751.46	751.46
15.0	4.3474	4.8321	4.8321	693.97	677.39	677.38
5.0	8.8456	8.8542	8.8542	561.14	550.95	550.94
5.	12.370	12.362	12.362	494.32	485.53	485.52
− 5.0	17.900	17.789	17.789	426.04	417.62	417.62
−15.0	45.322	45.922	43.923	274.66	264.42	26.43
−20.0	104.49	110.76	110.86	154.23	132.64	132.49
−20.051	106.79	119.77	121.34	151.31	122.96	121.34
−20.0521	106.85	121.28	unstable	151.25	121.44	unstable
−20.2567	127.78	unstable		127.81	unstable	
−20.2568	unstable			unstable		

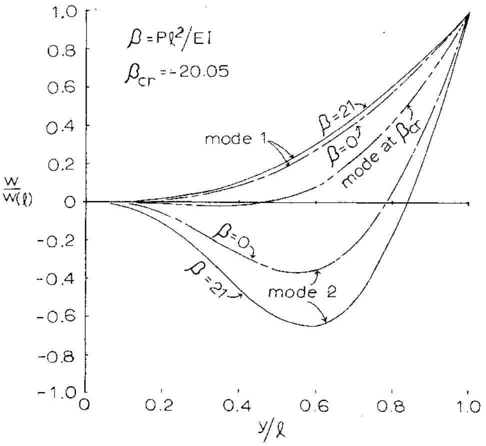

Fig. 2 First and second modes, various axial loads.

Solutions for various combinations of loading and boundary conditions, some of which are not to be found in the literature, have now been obtained from Eqs. (3) and (4).[4] The direct solution to the Beck problem is obtained when the load parameters and boundary conditions in Eq. (8) are specified such that they conform to the Beck problem. For a clamped end at $y=0$, set $P=2$. For the load, set β_1 through β_9 equal to zero and set $\beta_0=\beta_{10}$. This provides the discrete end load which remains tangent to the deflection curve.

Results

In Table 1, eigenvalues from the direct analytical solution are compared to eigenvalues from the exact solution for a wide range of the load parameter, $\beta_0=\beta_{10}=\beta=P\ell^2/EI$. Results from both a three- and a six-term solution are given in order to illustrate typical convergence for nonconservative and/or nonstationary systems. Convergence may occur from above or below as the accuracy of the solution is increased. The important point is that convergence does, in fact, occur just as Hamilton showed it would.[11]

Figure 1 shows curves of the eigenvalues vs load magnitude for the lowest four modes of the Beck problem. The curves for the direct and exact solutions are identical for the first and second modes. Circles denote discrete points from the exact solution for comparison to the curve (from only a six-term solution) for the third and fourth modes.

Finally, Fig. 2 shows a plot of the first and second normalized modes for a tension load, no load, and the critical compressive load. The severity of the change in configuration of both modes is readily apparent as the load magnitude is changed from tension to zero (at which free vibration of the cantilever beam occurs) and finally to the critical load (at which flutter of the cantilever beam-column occurs).

Conclusion

Application of Hamilton's law to a nonconservative, continuous system has been demonstrated. No variatonal principle, no D'Alembert's principle, no differential equation nor the theory thereof, and no work function (in the sense of the calculus of variations) were required or used.

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